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THIS JOURNAL IS DEDICATED TO THE FOLLOWING AIMS: (1) Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values. (2) To supply an additional medium for the publication of expository Mathematical articles. (3) To promote more scientific methods of teaching mathematics. (4) To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

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National Mathematics Magazine

The following are excerpts from letters recently received:

"I am enclosing my check for $2.00 for a year's subscription to National Mathematics Magazine. ...Louisiana State University has rendered a real service to mathematics and education not only in the South but throughout the Nation in supporting the Magazine." From H. M. Bacon, Stanford University.

"Almost from the first I have had our library subscribe to your Magazine. ...Appeals for help are coming from every direction from this and from other countries, but I feel that we should take the long view as well as the emergency one. ...I cannot promise aid for the endurance, but I am glad to help this year in any way and as long as I can.

"I am enclosing my check for $2.00 for a personal subscription and will have the library continue as usual." From C. E. Corbin, College of the Pacific.

"It is with great regret that I learn that the National Mathematics Magazine is having financial trouble. I know you have put in years of work and devotion to a journal, which is a credit not only to you but to the University and the South." From C. C. MacDuffee, Hunter College, New York City.

Such letters as these check well with the following statistics on the Magazine's growth in the last three years. An adequate index is our receipts for the central months January and Feburary.

	1939-40	*1940-41*	*1941-42*
January	$ 114.48	$ 94.39	$ 157.15
February	57.75	52.82	120.41*
Totals	$ 172.23	$ 147.21	$ 277.56

*This figure is as of February 25, 1942

Summing up: It would seem that in spite of present war conditions and of a more or less publicized prospect of drastic reduction in the Magazine budget the National Mathematics Magazine is still moving forward.

National Mathematics Magazine WILL CONTINUE TO GO FORWARD UNDER WHATEVER SPONSORSHIP IT MAY OPERATE.

S. T. Sanders.

Beyond Quadratics*

By Harold L. Dorwart
Washington and Jefferson College

1.

In many preparatory schools, a second course in Algebra is entitled, all inclusively, *Quadratics and Beyond.* The course usually discusses such topics as Progressions, Binomial Theorem, Permutations, Combinations, Probability, Theory of Equations, Determinants and Partial Fractions. The traditional so-called Higher Algebra courses have largely been concerned with further topics in Theory of Equations, including Symmetric Functions, Elimination, Resultants, Discriminants and Constructibility. But if one picks up such a book as Modern Higher Algebra by A. A. Albert of the University of Chicago, published in 1937, or an Introduction to Abstract Algebra by C. C. MacDuffee, published in 1940, he will find no explicit mention of any of the above topics, but instead will find the terms *modul, group, ring, field, matrix, transformation, valuation, structure, lattice, automorphism* and *ideal* occuring with startling frequency. And if he should then read in the Introduction to L'Algèbre Abstraite by Oystein Ore of Yale University, published in 1936, "L'Algèbre abstraite est une des disciplines les plus récentes de la mathématique moderne. Elle s'est développée principalement dans les dix ou quinze dernières années quoi qu'il existe des contributions importantes précédant cette période," and in Albert, "During the present century modern abstract algebra has become more and more important as a tool for research not only in other branches of mathematics but even in other sciences", he would doubtless be greatly surprised. The final blow, however, would probably come if he read in The Queen of the Sciences, by E. T. Bell, that there is not a single algebra but possibly as many as 1152 consistent systems of algebra!

Now is it possible to give the interested layman a non-technical account of the origin and content of modern algebra?. Probably this can be accomplished only to a degree. It is the present writer's conviction that the best way in which to make the attempt is to begin with a somewhat detailed study of a few of the fundamental concepts. The notion of a *field* is certainly one of these, and in the following

*Presented to the Allegheny Mountain Section of the Mathematical Association of America at Washington, Pa., October 25, 1941.

paragraphs emphasis will be placed on developing this concept, although several other concepts will take form incidently. First, however, for the purpose of background, a brief picture of our number system as the reader is probably familiar with it will be given.

2.

The simplest numbers are the positive integers $1,2,3,\cdots$, the so-called *natural* numbers. The fundamental nature of these numbers so impressed the great mathematician Leopold Kronecker that he once said, "the whole number was created by God, everything else is man's handiwork." The natural numbers are sufficient for the purpose of counting objects in any finite collection. But for direct measurements, such as determining the length or breadth of a book or a table top, they are not sufficient. Next came the *rational* numbers, numbers like $^1/_2$, $^5/_6$, and $^3/_1$ which can be expressed as the quotient of two integers. Since the denominator can be 1, the set of rational numbers evidently includes the set of integers as a sub-set. While the rational numbers are sufficient for direct simple measurements, they do not suffice for indirect measurements. For example, the length of the diagonal of a square whose side has been found by direct measurement to be one unit in length is $\sqrt{2}$ units, a number which was known to be *not rational* in Euclid's day. We now call such a number an *irrational* number.

For a long time after they first appeared, *negative* rational and irrational numbers were not regarded as legitimate members of the number family. René Descartes, the inventor of Analytical Geometry, in his famous book La Géométrie, published in 1637, refers to -5 as the *defect* of 5, and in speaking of the negative roots of an equation, calls them *false or less than nothing*. It was not until the widespread use of the measuring scale, such as is used on the ordinary house thermometer, gave them a pictorial representation, that they came into common use.

Although the number *zero* was used by the Babylonians, the Hindus, and the Mayas of Central America at approximately the beginning of the Christian era, it likewise did not come into general use in computations until comparatively recent times. Incidently, while speaking of zero, it might be well to remind the reader that the product of zero by any number and likewise the quotient of zero by any number not zero are both *defined* to be zero, but that the quotient of any number (including zero itself) by zero is left *undefined*.

The set of all positive and negative rational numbers, zero, and all positive and negative irrational numbers is called the set of *real*

numbers. The real numbers are sufficient for measurements, but are not sufficient to express the result of the operation of taking the square root of a negative real number. It is still customary to call such numbers as $\sqrt{-2}$ or $3+\sqrt{-5}$ *imaginary*, but the term is greatly out of date, since over a century ago it was shown that these numbers can be used for a graphical representation of vectors, (physical quantities having both magnitude and direction). Notation is simplified by defining i as a fixed root of the equation $x^2+1=0$, and a *complex* number is then defined as a number of the form $a+bi$, where a and b are real numbers. If $b=0$, the complex number is seen to be real, if $b\neq 0$, it is imaginary, and if $a=0$, $b\neq 0$, it is called a *pure imaginary*. Thus the set of complex numbers is seen to include the sets of reals and imaginaries. The relation of the sets to each other is shown in the following diagram.

Complex

Real | Imaginary

Rational | Irrational $\sqrt{3}$, $\sqrt[3]{2}$, π | Pure $3i$ | Mixed $4+2i$

Integral 3 | Fractional $4/5$

Real

Non-positive | Non-negative

Negative | Zero | Positive

3.

Without specifying the particular set, suppose now that a and b are any two members of one of the number sets mentioned above, which we call S, and let us make the following agreements:

If $a+b$ is in S, then S will be said to be *closed* with respect to addition.

If $a-b$ is in S, then S will be said to be *closed* with respect to subtraction.

If ab is in S, then S will be said to be *closed* with respect to multiplication.

If a/b is in S, then S will be said to be *closed* with respect to division.

A set closed with respect to addition and subtraction will be called a *modul*. A set closed with respect to addition, subtraction and multi-

plication will be called a *ring*. A set closed with respect to addition, subtraction, multiplication and division will be called a *field*.

It will now be seen at once that the set of all integers (positive integers, negartive integers and zero) forms a ring but not a field, but that the set of all rational numbers (division by zero excluded) forms a field, the *rational field*. Likewise the set of all real numbers forms the *real field*, and the set of all complex numbers forms the *complex field*.

One can easily see that the rational numbers are contained in every field, for if a is a number in a general field then $a/a=1$ is also in the field. Likewise $1+1=2$, $1\div 2=\frac{1}{2}$ etc. are in the field. The set of rational numbers forms the least field.

Now if we want to generalize and make *abstract* our notions of modul, ring and field (as well as obtain some new ones), all we have to do is (1) remove the restriction that our set be a number set, and (2) state explicitly as many as we plan to use of the assumptions or postulates that we implied when we used number sets. That is, let $S(a,b,\cdots)$ be any set, finite or infinite, of quantities or *elements*, subject only to the restriction that when we are confronted with a particular object of the universe we will be able to tell whether or not it belongs to S. The determination of the meaning of some or all of the rational operations of addition, subtraction, multiplication and division for any particular set, together with the formulation of the other postulates is a technical matter that need not concern us here. A set together with its associated rules for the combination of its elements is called a *system*, and algebra usually deals with systems which are closed with respect to one or more rational operations.

An illustration will probably now be helpful. The clockwise rotations of a plane configuration about a point in its plane form a set. Let the element a be the rotation through any particular angle, say 45°, and b be the rotation through some other angle such as 30°. The rotation obtained by applying *a followed* by b is usually defined in geometry to be the *product ab* of the two elements. Since this rotation is in the set, the set is closed with respect to multiplication as we have here defined it.

When we talk about the multiplication of numbers, the order of the factors in the product is immaterial, i. e. $3\cdot 4=4\cdot 3$. This is expressed by saying that ordinary multiplication of numbers is *commutative*, or that any two numbers obey the commutative law or postulate for multiplication. Is multiplication as we have defined it for the set of rotations commutative? A moments reflection will show that it is and that hence our illustration comprises a commutative system.*

*Actually an example of an important type of system known as a (commutative) *group*, whose position is somewhat intermediate between the modul and the ring.

It has now probably occurred to the reader that for certain sets and definitions of multiplication, the commutative property may not hold. This is actually the case. Vector multiplication and matrix multiplication are two such examples which are frequently used by the physicist. For vectors $a \cdot b = -b \cdot a$, and for matrices, $a \cdot b$ bears no relation to $b \cdot a$. Lest this situation upset the reader too much, we hasten to remind him that it has its analogue in ordinary plane geometry where the rejection of one of the postulates of Euclid gives rise to a non-Euclidean form of geometry.

4.

We are now ready to examine an important case where the concept of number field has brought order out of chaos and has indirectly been responsible for the invention of an important generalization of number.

An *algebraic* number is defined to be any number which satisfies an equation of the form

$$a_0x^n + a_1x^{n-1} + \cdots\cdots\cdots\cdots + a_{n-1}x + a_n = 0,$$

in which n is a positive integer and $a_0, a_1, \cdots, a_{n-1}, a_n$ is a set of integers, positive, negative or zero, except that $a_0 \neq 0$. Any number not algebraic is called *transcendental.* The number π is the familiar example of this type of number.* This classification of numbers cuts across the one given earlier in that both algebraic and transcendental numbers can be real or imaginary. Now how can we further classify algebraic numbers? The idea of a field will do this for us very nicely.

To be concrete, let us consider the set of numbers of the form

$$a + b\sqrt{-5} = a + bi\sqrt{5}$$

where a and b are rational numbers. These numbers are algebraic since second degree equations which they satisfy can quickly be found. The reader who remembers from elementary algebra the rules for operating with complex numbers, can quickly verify that the sum, difference, product and quotient of any two numbers of this set is of the same form as the general number and that hence the set forms a field. This is an example of a quadratic imaginary field. There are also quadratic real fields, cubic fields, etc., and thus algebraic numbers can be classified.

The particular example above was chosen because of a further fact. In the rational field, we have *unique prime decomposition* of

*Incidentally, the proof that π is transcendental, first given by Lindemann in 1882, is the keystone in the mathematical proof that the circle cannot be "squared" by ruler and compass alone.

integers, i. e. corresponding to any given integer, there is only one set of prime integral factors, a prime integer being one divisible only by plus or minus itself and plus or minus one. Thus while $12=3\cdot 4=2\cdot 6$, 4 and 6 are not prime and the *unique prime* decomposition of 12 is $2\cdot 2\cdot 3$.

How about decomposition in primes in an arbitrary field such as the quadratic imaginary field considered above? By now the reader is probably on his guard, but less than one hundred years ago, many able mathematicians made serious blunders in their work by *taking it for granted* that prime decomposition was unique in every number field.

The old adage says that the exception proves the rule, but in the deductive reasoning of mathematics, it is just the opposite; the exception destroys the rule. That is, in the present case, if we can find one example of the failure of the unique prime decomposition law, then the law does not hold in general.

Such an example can be found in the field of numbers of the form $a+b\sqrt{-5}$, where a and b are ordinary rational numbers. A rigorous demonstration is out of the question here, but enough details can be given so that the reader may get a good idea of the procedure. To avoid complications, we will not give the definition of an algebraic integer, but will merely state that in the field under consideration an integer will occur only when a and b are ordinary positive or negative integers or zero. The number 9 is an integer in the field (we recall that the rationals are contained in every field), and we have

$$9=3\cdot 3=(2+\sqrt{-5})(2-\sqrt{-5}).$$

To show that these two different decompositions are both prime, we must show that the numbers 3, $2+\sqrt{-5}$, $2-\sqrt{-5}$ are primes in our field. Suppose that

$$2+\sqrt{-5}=(x+y\sqrt{-5})(z+w\sqrt{-5}),$$

where x, y, z, w are ordinary integers. Then

$$2-\sqrt{-5}=(x-y\sqrt{-5})(z-w\sqrt{-5}),*$$

and by multiplication

$$9=(x^2+5y^2)(z^2+5w^2).$$

We are now operating in the field of ordinary rational numbers where the only integral factors of 9 are ± 1, ± 3, ± 9. Since $x^2+5y^2=\pm 3$ is evidently not solvable in integers, x^2+5y^2 must be either ± 1 or ± 9

*It is easily shown that the product of the conjugates of two complex numbers is the conjugate of their product. L. E. Dickson, *Elementary Theory of Equations*, p. 22.

and $z^2+5w^2=\pm 9$ or ± 1. Hence one of the assumed factors of $2+\sqrt{-5}$ is ± 1 and $2+\sqrt{-5}$ is prime. Likewise for $2-\sqrt{-5}$, and in similar fashion we can show that 3 is a prime, and that hence we have two distinct prime decompositions for the integer 9 in the chosen field.

As implied earlier, the attempts to remedy this failure led to the invention of new entities, the *ideal numbers*, for which the unique prime decomposition law holds. Although their invention is one of the great achievements of modern mathematics, it will not be possible to discuss them here.

5.

Two questions are probably now uppermost in the reader's mind. What *is* algebra in the modern sense? and why are mathematicians interested in it? Answers to these questions are contained in the two following quotations. The first is a characterization "ventured" by Saunders MacLane in a recent issue of the American Mathematical Monthly. "Algebra tends to the study of the explicit structure of postulationally defined systems closed with respect to one or more rational operations." Each word in this elegant and relatively brief characterization has been carefully selected for its precise technical meaning, and while it is hoped that the statement will now convey some meaning to the reader, he should not expect to get the full meaning without considerable study and thought.*

The second quotation is from the monograph by Ore mentioned at the beginning of this article. "On doit chercher les causes de l'intérêt toujours croissant que l'on a témoigné à cette partie de la mathématique dans la beauté des résultats, dans le charactère complet des solutions de plusieurs questions générales ainsi que dans des relations étroites de l'algèbre abstraite avec d'autres parties de la mathématique moderne."

*A Survey of Modern Algebra by Garrett Birkhoff and Saunders MacLane, published since this article was written, contains the statement, "One may even describe abstract algebra as the study of those properties of algebraic systems which are preserved under isomorphism."

On Integers of the Form $\dfrac{2^{p-1}-1}{p}$

By Adolph Goodman
University of Cincinnati

It is known from Fermat's theorem that if p is an odd prime, then $N=(2^{p-1}-1)/p$ is an integer. Uspensky and Heaslet* give the problem of proving that N is the square of an integer only for $p=3$, and $p=7$. In his class in number theory, Dr. I. A. Barnett suggested the problem of determining all p's such that N is the nth power of an integer. For completeness, we will treat both problems.

Theorem—*If p is an odd prime, then $N=(2^{p-1}-1)/p$ is the square of an integer only for $p=3$ and $p=7$, and N is the nth power of an integer $(n>2)$ only for $p=3$.*

Suppose in fact that $N=\propto^n$. As p is odd,

$$K=\frac{p-1}{2}$$

is an integer, and we can write

$$(2^K-1)(2^K+1)=p\propto^n. \tag{1}$$

Let $u=2^K-1$, and $v=2^K+1$. Then $v-u=2$, so that the greatest common divisor of u and v is equal to or less than 2. But if $K>0$, u, and v are both odd and hence they are relatively prime. Thus if q is a prime factor of $\propto$ which is also a factor of u, it cannot be a factor of v, and therefore q^n is a factor of u.

Let β^n be the largest nth power contained in u, and γ^n be the largest nth power contained in v. Then as p is a prime, there are only two cases to consider.

Case I	*Case II*
$(2)\begin{cases} u=2^k-1=p\beta^n \\ v=2^k+1=\gamma^n, \end{cases}$	$(3)\begin{cases} u=2^k-1=\beta^n \\ v=2^k+1=p\gamma^n. \end{cases}$

**Elementary Number Theory, McGraw-Hill, 1939. p.* 151.

In Case I, $2^k = \gamma^n - 1$, and since $n > 1$, γ must be odd. Now if n is odd then

$$2^k = (\gamma - 1)(\gamma^{n-1} + \gamma^{n-2} + \cdots + \gamma + 1)$$

and both factors on the right must be powers of 2. But the second factor is odd and greater than one for every positive integer γ.

If n is even, then $2^k = (\gamma^{n/2} - 1)(\gamma^{n/2} + 1)$, so that $\gamma^{n/2} - 1$ and $\gamma^{n/2} + 1$ must be powers of 2 differing by 2. Therefore $\gamma^{n/2} - 1 = 2$, $\gamma^{n/2} + 1 = 4$, which yields the solution $\gamma^{n/2} = 3$, $\gamma = 3$, $n = 2$, $K = 3$, $p = 7$.

In Case II, $2^k = \beta^n + 1$. This has the obvious solution $\beta = 1$, $K = 1$, whence $p = 3$ with n arbitrary furnishes a solution of the problem. Now let $\beta > 1$. Clearly β must be odd. If n is odd, then

$$\begin{aligned} 2^k &= (\beta + 1)(\beta^{n-1} - \beta^{n-2} + \beta^{n-3} - \cdots - \beta + 1) \\ &= (\beta + 1)[\beta^{n-2}(\beta - 1) + \beta^{n-4}(\beta - 1) + \cdots + \beta(\beta - 1) + 1]. \end{aligned}$$

If $\beta > 1$, then the second factor on the right is odd and greater than one, whereas it must be a power of two.

Now if n is even, then $2^k - 2 = \beta^n - 1 = (\beta^{n/2} - 1)(\beta^{n/2} + 1)$, and the right member is different from zero and is divisible by 4, while the left member is not.

Hence the only solutions of the problem are ($p = 7$, $n = 2$) and ($p = 3$, n artibrary).

Humanism and History of Mathematics

Edited by
G. Waldo Dunnington

U. G. Mitchell, 1872-1942

Professor Ulysses Grant Mitchell, from 1931 until a year ago head of the mathematics department of the University of Kansas, died on January 1, 1942, at his home in Lawrence; funeral services were conducted January 3 from the Congregational Church of that city. Professor Mitchell was born November 26, 1872, at Nashua, Iowa, and lived there until the age of twelve; in the spring of 1885 he went to Kansas in a prairie schooner. The family settled near Peabody, on the site where the "Watchhorn" oil well is now located.

Of the Peabody years in grade and high school, he once said: "I herded 160 head of cattle at a time when I was 13, and during my high school days earned my way working in a grocery store." An old army officer, then superintendent of the Peabody high school, instilled in young Ulysses his first interest in mathematics. As a senior, he taught the superintendent's classes during the latter's illness of a month.

After high school graduation, Mr. Mitchell taught a country school for three years, having as many as 50 boys and girls for pupils. Following this he attended the University of Nebraska a year, and then a teachers college, where he taught mathematics and German besides going ahead with his school work. He was married at this time, having met the future Mrs. Mitchell in the eighth grade, and served as superintendent of schools in Hillsboro, Kansas from 1898-1902. Mr. Mitchell was superintendent of schools in Lyons, Kansas schools the following two years.

Entering the University of Kansas in 1904, he earned the A. B. in 1906 and the M. A. in 1907. In 1905 he taught the University's first course in American Government. In the fall of 1906 he was appointed instructor in mathematics and rose to a full professorship and the headship of the department. His doctorate of philosophy in

mathematics, with history as minor, was conferred by Princeton in 1910.

Dr. Mitchell's chief mathematical fields of research were projective geometry and theory of numbers. His publications were numerous and of a quality eliciting praise from his colleagues. It is to be hoped that a bibliography of them will be prepared and published. In addition to this, he made a reputation as an historian of mathematics. His special interest in mathematical recreations is evidenced by this statement: "I have never played a game in my life that I didn't like." As a country school boy young Mitchell made quite a name locally as a checker player, but in later years chess and bridge were his favorite games. His ability as an organizer was notable, as shown by his great interest in problems connected with the teaching of mathematics at all levels and other related curricular matters in administration. His special hobby was the collecting of old mathematical works; these rare volumes in his private library number several hundred and for the most part are written in Latin, German and French. Many of them were purchased when, taking a leave of absence for the spring semester and summer of 1932, he and Mrs. Mitchell toured points of mathematical interest in Europe, inspected scientific museums as well as manuscript collections and attended the International Congress of Mathematicians in Zurich. Dr. Mitchell served as associate editor of the *American Mathematical Monthly* 1914-19.

During a long period of years these learned bodies claimed the attention and activity of Professor Mitchell: American Association for the Advancement of Science, American Mathematical Society, Mathematical Association of America, History of Science Society, and the Kansas Academy of Science. He was prominent in the councils of the Congregational Church and served on the Y. M. C. A. board at the University.

The University of Kansas has never known a harder working faculty committeeman than Dr. Mitchell. For ten years he served as chairman of the committee on "Relations with Other Educational Institutions." He worked on the Junior College Committee and the committee that organized the Memorial Union in 1927-28, as well as holding the chairmanship of the University Survey Committee. When the University of Kansas celebrated its Diamond Jubilee in 1939, it was only natural that Professor Mitchell should have been made chairman of the committee in charge.

Dr. Mitchell's deep devotion to Phi Beta Kappa and its ideals, along with his active support of it, deserves special mention here. This is the record: chairman, North Central district; president, Kansas

chapter, 1924-26; vice-president, 1922-23; secretary, 1913-14; treasurer, 1927-30; investigator of colleges and universities, Committee on Qualifications, 1931-34.

The personality of Professor Mitchell was characterized by a genial, subdued and affable nature. When stirred by some injustice or dishonesty, he spoke with unexpected directness and vigor. He had a naive sense of humor, and was easily approachable. He took apparent pleasure in helping others, even in little things. Withal he was a likeable person, a true scholar and a good friend. His life was one of goodwill and service to his fellow men, and to institutions in which he believed.

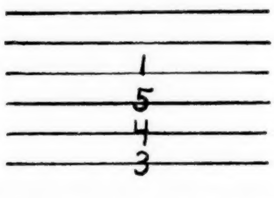
G. WALDO DUNNINGTON.

Early English Arithmetics

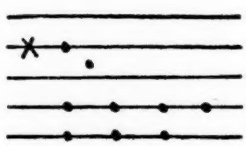
By E. R. Sleight
Albion College, Albion, Michigan

(Concluded from January, 1942, issue)

Methods of performing the fundamental operations by means of mechanical devises were known long before our modern system was devised. To all such devises the word "Abacus" was applied. They were of three standard types, the dust board, the table with counters fastened to lines and the table with loose counters. The second dialogue in Record's *Grounde of Artes* begins with a statement from the Master. "Now that you have learned Arithmetic with the pen, you shall see the same art in Counters. The system consists of horizontal lines. "The fyrste line is the place of unites, and every counter set in that line betokeneth but one: the seconde line is the place of 10, and every counter there stands for 10: the third line the place of hundreds, and so forth. Following this explanation of the system, the Scholar is asked "If you do understande it, then how shall you set 1543? The number is placed on the counter board in this manner.

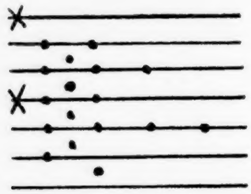


Master: You have set the places truly, but your figures be not meet for this use: for the meetest figure in this behalf is the figure of the counter round, as you see here, where I have expressed that same summe."

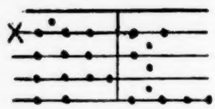


Placing the one counter for 500 in the space above the third line puzzled the Scholar until the Master explained that a counter placed in

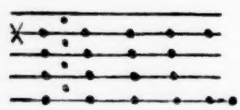
any void space is to be multiplied by the value of the line beneath, as in this case "because the *numerator is 5, it must be set in a void space: and because the *denominator is a hundred, I knowe that his place is the void space above hundreds." To aid in reading numbers a crosse is placed on the line indicating thousands, as well as on the line for millions. When the Scholar interpreted the following to represent 287965 he was commended by the Master because of his ability to read numbers after such a brief explanation.



The fundamental operations by use of the counters were explained by the Master, beginning with addition. "Therefore when you will adde two summes, you shall first set down one of them, and then by it draw a line crosse the other lines. And afterward set downe the other sum, so that that line may be between them, as if you would adde 2659 and 8342. You must set your summes as you see here."

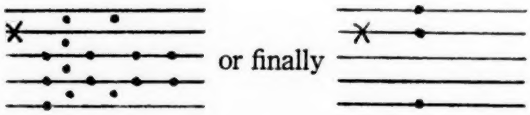


Each number is said to occupy a †room. The process begins with the units, followed by the tens, etc. In this manner the following result is obtained:



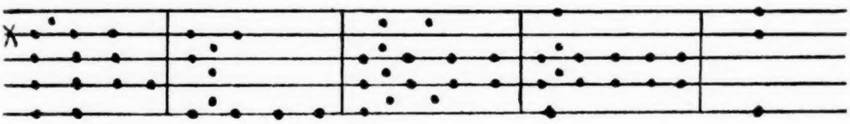
But since two dots between two lines is equivalent to one on the upper of the two lines, and five on a line make one in the space above it, the result may be expressed in this manner:

*Not to be confused with a fraction.
*roume.

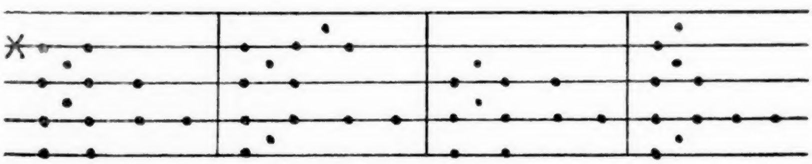


which is equivalent to 11001, the sum of the two given numbers.

By introducing other rooms, the entire process may be pictured in this manner.

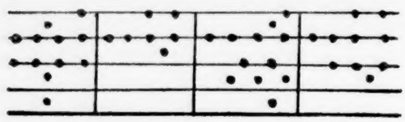


In the language of the Master "and so you see the whole summe to be 11001; but if you have more than two summes to adde, first adde two of them, and then adde the third." At this stage the Scholar becomes anxious, and suggests "Nowe I think best that you passe forth to Subtraction." To which the Master responds "There is the same proof here that there is in Addition by the pen, I meane Subtraction, but I will first teach you the Art of Subtraction." For this purpose it is proposed to subtract 2892 from 8746. The Master suggests that it is better to "set the lesser number first, then shall I begin to subtract the greatest numbers first (contrary to the use of the Pen)." Noting that the larger Number contains 8 of the thousands denomination while the smaller contains only 2, subtraction leaves 6. This result is placed in the fourth room, while the third is used to represent 892, the remainder of the smaller number after removing the 2 thousand. The original numbers, together with the changes thus noted are indicated in this manner.



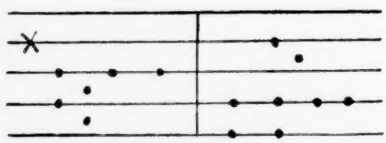
In attempting to operate on the hundreds digit, the Master calls attention to the fact that in the "first summe I find 8, and in the second but 7." He then explains the "borrowing" process, and uses more rooms to indicate how the subtraction may be performed. Rewriting the number in the third room above just as it is, and replacing one of the dots on the fourth line of the number in room four by two

dots in the space below, thus reducing the 1000 to hundreds, we have the first two rooms in the figure below. The third room represents the original subterhend after removing the 2800, while in the fourth is found the difference thus far. After removing the "articles or tennes" and the "unities" the final difference is here represented.

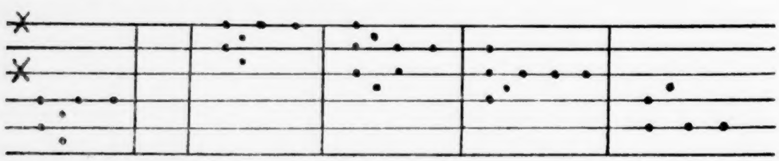


In multiplication the numbers are placed in rooms just as in the other operations, the multiplier being placed in the first room. The

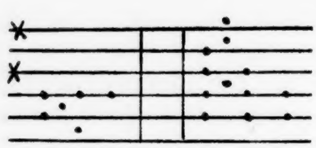
Master says "Take this example: I would multiply 1542 by 365, therefore I set the numbers thus.



Then first I begin at the 1000 in the highest room." The product of 365 and 1000 is then represented in the third room, while in successive rooms are found the products of 365 times 500, 365 times 40, and 365 times 5. As each partial product is placed in its room, the corresponding digit in the multiplicand is omitted until finally it disappears. The entire process is thus represented.

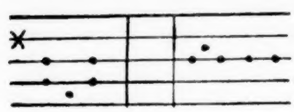


Combining the partial products as represented in the last four rooms, there appears:

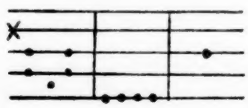


The product is then found to be 562830. "Whereby you see that 1542 (which is the number of years since Christs incarnation) being multiplied by 365 (which is the number of daies in one year) doth amount unto 562830, which declareth the number of daies since Christs incarnation unto the end of 1542 years."

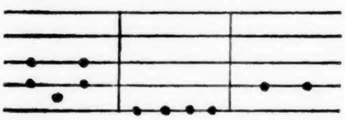
According to the Master, the surest way to prove multiplication is by division. "Therefore would I over passe it till I have taught you the art of division." To illustrate the process 900 is divided by 225. For this purpose three rooms are used. In the first is placed the divisor, while in the second and third respectively the quotient and the dividend are found. "Then begin I at the highest line of the dividend, and seek how often I may have the divisor therein, and that may



I do four times." Four times two is then subtracted from the nine, leaving:



The four dots in the middle room represents the quotient. "But now must I take the rest of the divisor as often out of the remainder therefore come I to the second line of the divisor, saying 2 four times make 8, take 8 from 10 and there resteth 2, thus:"



It will be observed that the 2 to which the Master refers in the last operation represents 2 times 10. "Then come I to the lowest number which is 5, and multiply it 4 times, and take it from 20 there remaineth nothing, so I see my quotient to be 4."

Before completing the Dialogue on Counters, the Master calls attention to a system "wherein are both pence, shillings, and pounds, proceeding, by no grounded reason, but only by a form, and that diversely by divers men; for merchants use one form and auditors use another." In the mind of the Master these processes do not seem to be very necessary, for after a very brief statement concerning them,

he concludes in this manner, "But in this thing, you shall take this for sufficient, and the rest you shall observe as you may see by the working of each sort."

In Record's discussion of fractions we find some very familiar terms,—proper, improper and mixed numbers. Not so with the notation, and manner of expressing the operations. Thus $^3/_4$ of $^2/_3$ of $^1/_2$ is written $\begin{smallmatrix}3&2&1\\4&3&2\end{smallmatrix}$. The author points out that "some others make lines and add words of distinction thus: $^3/_4$ of $^2/_3$ of $^1/_2$. Still others express them in slope form $_{\frac{1}{2}}\ \frac{2}{3}\ ^{\frac{3}{4}}$, to distinguish them from fractions of one whole number."

Under reduction of fractions several divisions are pointed out.

1. Fractions must be reduced to a common denominator.
2. Fractions of fractions must be reduced to a single fraction.
3. Improper fractions may be separated.
4. Fractions may be reduced to lowest terms.
5. Numerator and denominator may be multiplied by a certain number so as to make the denominator a certain number.

In reducing fractions to a commond denominator, he does not recognize the possibility of a least common multiple, but rather the common denominator used by Record is the product of all of the denominators. Exceptions are made when one denominator is a factor of the other, or when all of the denominators are factors of some number less than their product.

Use is made of the highest common factor in the process of reducing fractions to lowest terms, the highest common factor being determined by the method of division. This method seems to disturb the Scholar for he asks "But is there no other way to work this reduction?" The Master replies "with more ease for a learner, you shall mediate the Numerator and Denominator, if the numbers be even. But if they be odd (though it be but one of them) then must you divide them by 3, 5, 7, or some such number." The form to be used is indicated, after which the Master continues "As for example I would reduce $^{288}/_{576}$ into his least terms or value." Step by step the process is performed, the result written is this manner.

288	144	72	36	18	9	3	1
576	288	144	72	36	18	6	2

The Master suggests some short cuts, for which "the ability in knowledge is got by exercise." By observation 12 is discovered to be

a divisor of both numerator and denominator of $^{60}/_{96}$. Also "When your fraction hath any cyphers in the first places of both terms, then may you by casting away the cyphers make an easier reduction as thus $^{300}/_{400}$, here take away the cyphers, and it will be $^{3}/_{4}$."

Scholar: "And so if I have $^{400}/_{650}$ then will I have $^{4}/_{65}$."

Master: "You are deceived, for you took away more cyphers from the Numerator than you did from the Denominator, which one may not do." There is no suggestion that this short cut is merely dividing numerator and denominator by the same number.

Occasionally the master seems at a loss for an explanation that is entirely satisfactory to the Scholar as in this question concerning multiplication, "But I see not how the third number in this example of fractions can contain either of the former (as it happens in whole numbers) seeing it is lesser than either of them." The Master responded "No matter if you can not see that thing which is not possible to be seen of any man." Then follows a lengthy discussion in an attempt to make clear the seeming inconsistency, ending with "Then seeing that a fraction is less than one, if I multiply a fraction by another fraction it followeth that I do take the first fraction less than once, and therefore the sum amounteth must needs be less than the first fraction."

In multiplication of fractions, while cancellation is not used as a process, it is recognized in the following discussion concerning the multiplication of a whole number by a fraction. "If it happens the Denominator to be such a number as may evenly be divided by the said whole number, then divided thereby, and set the quotient of that division for the former Denominator, but retain the Numerator."

Scholar: Then I take this example $^{7}/_{20}$ to be multiplied by 5; and because 5 will exactly divide 20, therefore I take the quotient of the division, which is 4, and set it instead of 20, and so the fraction will be $^{7}/_{4}$, that is $1^{3}/_{4}$."

As an aid to subtraction, a St. Andrews Cross is used. Thus in subtracting $3^{1}/_{3}$ from 6, the result is displayed in this manner:

$$\begin{array}{ccc} & 8 & \\ \frac{10}{3} & \times & \frac{18}{3}, \\ & 3 & \end{array}$$

in which the numerator and denominator of the difference appear in the upper and lower vertical angles. In multiplication a similar plan is used. When it is proposed to multiply $^{3}/_{5}$ by $^{5}/_{12}$, the result is thus indicated

$$\begin{array}{ccc} & 15 & \\ ^{3}/_{5} & & ^{5}/_{12} \\ & 60 & \end{array}$$

the cross being omitted.

The method of inverting the divisor and proceeding as in multiplication is used indirectly. "Then shall I multiply the Numerator of the Dividend by the Denominator of the Divisor, and the result I must put for the new Numerator. Again I shall multiply the Denominator of the Dividend by the Numerator of the Divisor and that number I must put for the new Denominator." The operation of division is pictured in the same manner as the operation of multiplication, as is indicated by the example, "to divide $^5/_8$ by $^2/_6$," the solution of which appears thus:

$$\begin{array}{ccc} & 30 & \\ ^5/_8 & & ^2/_6 \\ & 16 & \end{array}$$

Very frequently use is made of coins to explain an operation. In division of fractions the Scholar is made to say "If I would divide $^2/_3$ by $^3/_4$, I must set the numbers one against the other, thus, $^2/_3$ by $^3/_4$ and then make another line for the quotient at some good distance where I may set the numbers of the quotient, as soon as any of them is multiplied. So then as soon as I have multiplied 2 by 4, which makes 8, I shall set that 8 over the line thus $\underline{\ 8\ }$, and then multiply 3 times 3, which yieldeth 9; and that 9 must be set under the same line and then will the whole Quotient appeare $^8/_9$, wherein it appeareth that $^2/_3$ is in proportion to $^1/_4$ as 8 is to 9, but how may I perceive that?"

Master: "This example may be declared in common coins, as in a shilling if 12 pence of which $^2/_3$ maketh 8 pence and $^3/_4$ doth make 9 pence, and so you may safely see that their proportions doe agree."

The second part "touching fractions" closes with the application of the Golden Rule to correct certain errors in the Statute Books. According to the Master, "If you examine the Statute you shall find many summes false. I would, therefore, suggest that all Gentlemen and other students of the Laws would not neglect the Art of Arithmetic, as unneedful to their studies. Wherefore to encourage them I will exhibit a table of that part of the Statutes in two Columns, and in a third Column I will add a correction of those errors which have crept into it. Here followeth the table."

THE PRICE OF A QUARTER OF WHEAT		THE WEIGHT OF A FARTHING WHITE LOAFE BY THE STATUTE BOOKS			THE CORRECTION BY JUST ASSISE		
s	*d*	£	*s*	*d*	£	*s*	*d*
1	0	6	16	0	6	16	0
1	6	4	10	8	4	10	8
2	0	2	8	0	*3	8	0
2	6	2	14	$4\frac{1}{5}$	2	14	$4\frac{4}{5}$
3	0	2	8	0	2	5	4
3	6	2	2	0	1	18	$10\frac{2}{7}$
4	0	1	16	0	1	14	0
4	6	1	10	0	1	10	$2\frac{2}{3}$
5	0	1	8	$2\frac{1}{2}$	1	7	$2\frac{2}{5}$
5	6	1	4	$8\frac{1}{4}$	1	4	$8\frac{3}{11}$
6	0	1	2	8	1	2	8
6	6	0	16	11	1	0	$11\frac{1}{13}$
7	0	0	19	1	0	19	$5\frac{1}{7}$
7	6	0	18	$1\frac{1}{2}$	0	18	$1\frac{2}{3}$
8	0	0	7	0	0	17	0
8	6	0	16	0	0	16	0
9	0	0	15	$0\frac{1}{4}$	0	15	$1\frac{1}{3}$
9	6	0	14	$0\frac{3}{4}$	0	14	$3\frac{15}{19}$
10	0	0	13	$7\frac{1}{2}$	0	13	$7\frac{2}{5}$
10	6	0	12	$11\frac{1}{2}$	0	12	$11\frac{3}{7}$
11	0	0	12	$4\frac{1}{4}$	0	12	$4\frac{4}{11}$
11	6	0	11	10	0	11	$9\frac{21}{23}$
12	0	0	11	4	0	11	4

*This is evidently an error and should be 2.

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The Teacher's Department

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The Use of Index Numbers in Evaluation

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The most generally-used means for quantitative evaluation of instruction is that of the "average" or measure of central tendency determined from the test scores of the class. For this average, either the median or mean is used. The averages of several classes may be compared to appraise the instruction of the unit tested. In order, however, to evaluate instruction of the specific topics of a unit, further analysis is necessary. It is the purpose of this paper to consider a method for such further analysis.

As has often been stated by practical men, there is little point in collecting statistical data, unless its analysis is to be used to control the source of data. The source of data used in any testing program is in the complex phenomena of interrelation of test, student, and instructor. The statistics obtained in a testing program should therefore have characteristics which permit the facts learned to be used in changing the source of data. This entails improvement of the tests and the instruction, and, through the instructor, the increasing of the student's knowledge. In giving directions for attempted improvement, it is necessary that a study be made of the answers, both to specific questions, and to groups of questions concerning a particular area of instuctional materials. When the program of evaluation is able to state that this or that test question does not measure the knowledge expected, then and only then can improvement of the testing program be effectively carried out. When the program of evaluation is able to point out to an instructor that in such and such a topic, recognized as important by all instructors, his classes have shown consistently less knowledge than other classes, then and only then can improvement of instruction be guided. When experiments with instructional ma-

terials are being conducted much can be learned as to the teachability of the topic, the extent to which instruction is significant, and the amount of drill necessary in specific topics, and in areas of instruction.

The result of a student's effort on a question reflects in part the type of instruction given him. This result is a total of effective components: explanation and its clarity, illustration and its effectiveness, and drill and its fruitfulness. The results on the test also reflect the significance of a question, the importance of the topic covered in respect to the multitudinous topics considered, and the understandibility of a question, as to the agreement of its wording with that of the type of problem basic in the drill work of the course. The result of each question reflects the ability of the student—his preparation, memory, understanding, skill, and his ability to transfer from a theory to a specific problem.

We have found that the analysis of the statistical material collected from the results of tests will separate these three very complex and interrelated variables: instructor, student, and question. For this purpose we will discuss the methods, their properties and their uses.

The instrument we have found effective in the analysis is the index number. This is calculated for each group of students by means of the formula

$$I = \frac{c-f}{c+f} \; *$$

where c is the number of students answering the question correctly, and f is the number giving incorrect answers. In determining the index number, an answer to a question is considered incorrect unless full credit is given in grading. The index number obviously has a range $+1$ to -1, $+1$ signifying that all students answered the question correctly, and -1 signifying that all failed to answer correctly. An index number of zero indicates that only half answered the question correctly. For the benefit of those who consider the ratio (P) of correct answers to the total number of answers, P and I are related by the formula $2P-1=I$. This relationship may be used to interpret the difference between index numbers of two groups of students.

The index numbers on a question calculated for the different groups are: class index number for the students in one class; instructor's index number for the students in all the classes of one instructor; department index number for the students in all sections of the class taught by the staff.

*J. S. Georges, NATIONAL MATHEMATICS MAGAZINE, 16: 94 (1941).

These different index numbers show particular characteristics and may be used to distinguish the three essential variables governing the index number on a question. When the same test is used with only a few changes, the index numbers may be compared over a period of several semesters. Study of these index numbers has shown several properties.

1. We consider the department index number. It has been found that the department index number on a question will be remarkably stable throughout the use of a test. At Wright Junior College, we use essentially the same set of tests throughout three semesters. The index numbers on one question might be as high as .9, and on another as low as $-.8$, but the average range of variation on a single question is .19 with a standard deviation of .11 and a probable error of .009. Hence the department index number may be said to be essentially a function of the one variable, the test question and its relation to the course plan. The department index number thus fixes an evaluation upon the question, and one can predict the department index number on the question during later semesters if the form, text and plan of the course remain the same. In fact, the department index number will remain unchanged unless a specific planned effort is made by the department staff in the instruction of the topic. Hence much can be inferred about the question, its place on the tests, and the position of the topic in the instructional materials of the course. If the question considered has an exceptionally low index number, it is examined for its importance in the course, and its relation to the subject matter. Or it may be examined for simplicity of wording and its agreement with the form of corresponding questions in the student's preparation. If the question is vital, this examination usually will tell what steps should be taken to improve the instruction, and if the question is found to be faulty, to improve or change the question.

Because of its stability, it may be said therefore, that the index number evaluates the question. When the question has been formulated carefully so as to require in answering a special type of learning, the index number will tell much in evaluating the degree of this type of knowledge required for the mastery of the topic. In this regard several important facts have been ascertained, made possible by careful planning of the aims and objectives of the course and divisioning of the tests according to the types of learning involved.* Examination of the test results have shown, for example, the status of the scientific notation in the course. Ninety percent of the students were able to read the meaning of a number (very large or very small) in the

*ibid· 95-97.

scientific notation and to transfer from ordinary notation to the scientific notation, showing that one definite objective of the course has been achieved. The students have been prepared in this particular for this modern age of astronomic numbers and precisions measurement. As might be expected, from the difference between an awareness of the significance of an idea and the ability to use the idea skillfully, only forty-six percent of the students were able to evaluate the product of two numbers and express it in scientific notation.

Evaluation of general aims is more complicated because of the difficulty of setting up a testing program which will measure these very general objectives, and because of the careful divisioning of the test questions into sections requiring different types of learning. Examination of these questions shows some interesting information with regard to success of students in the different types of learning. We recall that, as described in an earlier paper of this series,* the questions of Part I require recognition of the correct mathematical relation, concept or fact in terms of its significance. Some three hundred students during one semester have shown themselves 77% accurate in recognition of mathematical concepts and processes.

2. We now consider the index numbers of the classes of individual instructors, and compare them with each other and with the index number of the department.

Variation of the index number from instructor to instructor on any particular question was found to be large. For six instructors teaching the first half of one introductory mathematics course during a single semester, the average range of index numbers on any one question was .7, while for three semesters the average range was approximately 1. Investigation of the variation of index numbers of the different classes of a single instructor during the same semester showed that the index number on a particular question was more closely related to the instructor than might be expected, the average range being .18. There was no recognizable permanence of superiority of one class to another in regard to frequency of higher index numbers unless medians of the classes showed wide variation of ability. One class might maintain a higher average index number but the two classes would fluctuate with the instructor as his index number moved from top to bottom of the range for the staff.

Some instructors showed a remarkable stability of index numbers over the three semesters when the same battery of tests was used; in fact, the range of index numbers was comparable to that of the department in this case. Some instructors, teaching classes of widely

*Ruth M. Ballard, NATIONAL MATHEMATICS MAGAZINE, 16:154-155 (1941).

differing ability, had a wide range of index number variations. In one case, although the average ability of two classes was the same, the aggressiveness of one class demanded instruction of a different type from that of the other, thus yielding wide variation on almost all questions of the tests.

We have noted that the correlation between the average index numbers on a test and the median of the class has been high, in fact +.9. Absolute correlation is hardly to be expected because of two principal facts. First, when the classes are small the median class may have relatively low frequency and hence the median is affected by the choice of class boundaries. Second, teachers may vary in the giving of partial credit. This may raise the median but does not affect the index number.

3. We illustrate the whole analysis with some special cases, stating the question and the index numbers obtained from the different sections of the class.

EXAMPLE 1

"Complete the following statement:

If $y=f(x)=x^2-3x+2, f(k)=$ ...."

The index numbers of the whole department and the different sections obtained during the present semester are as follows:

Dept.	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
.24.	−.03	.81	.38	.32	.35	−.48	.50	−.03	.18

Note that the sections C, D, E and I exhibit index numbers near that of the department, whose index number .24 shows that 62% of all students understand the use of the symbol $f(k)$. Note that section B has shown superior average understanding of topic, 90% of the section answering the question correctly. Only a few of the students in section F understand the meaning of $f(k)$. Section F did much better on the next question (see Example 2) requiring equivalent knowledge but not in the form of symbolism.

EXAMPLE 2

"Consider the function $y=3x-4$ which is represented graphically by a straight line.

a) If $x=1$, $y=$ ....; $x=3$, $y=$ ....

b) Hence the line passes through the points P_1 (.........,.........) and P_2(.........,.........).

c) Plot the points and draw the line.

d) Using these two points draw lines representing Δy and Δx in your figure.

e) The slope of the line has the value............ ."

The index numbers for department and sections B and F are given as follows:

	Dept.	*B*	*F*
a)	.83	.87	.93
b)	.66	.38	.93
c)	.78	.56	.78
d)	.46	.75	.56
e)	.20	.12	.26

Section F showed up poorly in understanding the meaning of $f(k)$ but here they are able to use the same understanding in a specific numerical case. Obviously the instructor had not drilled the students in the formal process. This example shows also the principle of the survival of the fittest in the problem of finding the slope of a line joining two points. Thirty percent of the students fall by the wayside between the first hurdle and the last.

EXAMPLE 3

"Express the following numbers in scientific notation.

a) 800,000 = ... b) .0003 = ...

c) Their product expressed in scientific notation is............"

The index numbers are as follows:

	Dept.	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
a)	.82	.93	.91	.86	.81	.66	.66	.55	.87	1.00
b)	.87	.86	.91	.86	.94	1.00	.66	.91	.81	.81
c)	−.22	−.10	−.30	−.29	−.15	.03	−.67	−.45	−.03	−.25

On part (a) of the question only Section G shows significant divergence from department index number (difference being .27). Note the drop from .82 and .87 to −.22 and hence the difference in type of knowledge required. It is one thing to translate into scientific notation but quite another to use it. (Note that the product, 24×10^1 must be rewritten 2.4×10^2.) Section F shows itself to be consistently poor.

EXAMPLE 4

"On logarithmic paper (base 10) the graph of $y = 10x^2$ is a straight line with a) slope = ...

b) y intercept = ..."

Index numbers on this question:

	Dept.	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
a)	−.42	−.24	.39	−.91	−.57	−.37	−.25	−.45	−.75	.25
b)	−.55	−.58	.13	−.91	−.57	.20	−.75	−1.00	−.84	.69

This question obviously covers a part of the course not given the emphasis in instruction comparable to its difficulty. Perhaps we are requiring too high a degree of complete understanding of one small part of a first course in analysis. Only 25% of the students were able to answer this question correctly. Sections B and I show that a fair proportion of the students were able to retain this portion of the material. Section C shows the topic was retained by a bare 5% of the class. Tests in previous semesters have shown that 60% of the students understand that the graphs of power functions on logarithmic coordinates are straight lines.

The general conclusions are that the index numbers furnish a panoramic view of effective instruction. The variation of an index number due to unexplainable causes is slight, about .2. Variation above .3 or .4 thus suggests the possibility of the determination of its causes. The index number will show therefore the area in which the instructor should put forth special effort to improve instruction.

In the public junior college it is important that definite checks be kept on the preparation given by the various sections of the same course. This check is supplied by the index number. The index number may be used therefore to improve the testing program and to direct the improvement of instruction.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, L. S. U., Baton Rouge, Louisiana.

SOLUTIONS

No. 412. Proposed by *S. B. Townes*, University of Oklahoma.

If p is a prime of the form $4n+1$ and K is any positive integer, show:

(a) There exists a representation $x^2+y^2+pz^2$ of Kp if and only if K cannot be written in the form $4^r(8s+7)$, $r\geqq 0$, $s\geqq 0$.

(b) There exists a representation $x^2+y^2+2pz^2$ of Kp if and only if K cannot be written in the form $2\cdot 4^r(8s+7)$, $r\geqq 0$, $s\geqq 0$.

Solution by the *Proposer*.

For every prime p of the form $4k+1$ there exists* a unique pair of integers, a, b, such that

$$a^2+b^2=p.$$

If and only if K is not of the form $4^r(8s+7)$ there exists* integers u, v, w, such that

$$u^2+v^2+w^2=K.$$

Then $\quad Kp=(u^2+v^2)(a^2+b^2)+pw^2=(ua+bv)^2+(ub-va)^2+pw^2.$

This is the proposed result (a).

If and only if $K/2$ is an integer not of the form $4^r(8s+7)$ we have as before

$$K/2=u^2+v^2+w^2.$$

*See L. E. Dickson: *Modern Elementary Theory of Numbers.*

Then $$Kp=2(u^2+v^2)(a^2+b^2)+2pw^2$$
$$=(ua+ub+vb-va)^2+(ua-ub+va+vb)^2+2pw^2$$
as required for (b).

No. 414. Proposed by *E. P. Starke*, Rutgers University.

Let α and β be the roots of $t^2-at-1=0$ and let $f(x)=x/(1-ax-x^2)$. Show that $f^{(n)}(0)=n!(\alpha^n-\beta^n)/(\alpha-\beta)$.

Solution by the *Proposer*.

Since $t^2-at-1=(t-\alpha)(t-\beta)$ we have
$$x^2+ax-1=(x+\alpha)(x+\beta).$$
Using the method of partial fractions we obtain
$$\frac{-x}{(x+\alpha)(x+\beta)}=\frac{\alpha}{\beta-\alpha}(x+\alpha)^{-1}+\frac{\beta}{\alpha-\beta}(x+\beta)^{-1}.$$
Thus the required nth derivative is evidently
$$(-1)^n n![\alpha(x+\alpha)^{-n-1}/(\beta-\alpha)+\beta(x+\beta)^{-n-1}/(\alpha-\beta)].$$
For $x=0$, this gives
$$f^{(n)}(0)=(-1)^n n!\,[-\alpha^{-n}+\beta^{-n}]/(\alpha-\beta)$$
$$=(-1)^n n!\,[\alpha^n-\beta^n]/\alpha^n\beta^n(\alpha-\beta).$$
Since $\alpha\beta=-1$, this is the desired result.

It is easy to show that this nth derivative, for $x=0$, equals
$$n!\left\{a^{n-1}+\binom{n-2}{1}a^{n-3}+\binom{n-3}{2}a^{n-5}+\binom{n-4}{3}a^{n-7}+\cdots\right\}.$$
These numbers, apart from the factor $n!$ are also the numerators and denominators of the successive convergents to the continued fraction
$$\frac{1}{a+}\ \frac{1}{a+}\ \frac{1}{a+}\ \frac{1}{a+}\cdots$$
which represents the positive root of $x^2+ax-1=0$.

No. 426. Proposed by *E. P. Starke*, Rutgers University.

P is a point equidistant from two perpendicular lines, *m* and *n*. Find the locus of a point whose distance from *P* equals the sum of its distances from *m* and *n*.

Solution by *M. S. Robertson*, Rutgers University.

Choose the lines m and n for coordinate axes and let P have the coordinates (a, a), $a>0$. Then the condition of the problem gives the relation

$$|x|+|y|=\sqrt{(x-a)^2+(y-a)^2},$$

where all distances are taken in an undirected sense since no coordinate system is implied in the statement of the problem. According as (x,y) is in the first or third, or is in the second or fourth quadrants, this equation becomes

(1) $$(x+a)(y+a)=2a^2 \quad \text{or}$$

(2) $$(x-a)(y-a)=0,$$

respectively. Since $x=a$ does not enter the second quadrant nor $y=a$ the fourth, the required locus consists of the parts of the hyperbola $(x+a)(y+a)=2a^2$ which lie in the first and third quadrants, together with the half-lines $y=a$ in the second quadrant and $x=a$ in the fourth.

Also solved by *Walter B. Clarke*, and *Robert W. Sutton*.

Editor's Note. A number of replies were received in which the condition of the problem was given as $x+y=[(x-a)^2+(y-a)^2]^{\frac{1}{2}}$. Of course, $x+y$ gives the algebraic sum of the directed distances of P from m and n, where it is presupposed that a positive sense has been agreed upon for such distances. Even if the problem had implied the use of directed distances it is difficult to imagine a reasonable definition for the sense of the distance of (x,y) *from* P which would make this equation (without squaring) true in both the first and third quadrants.

We have here a difficult point in the teaching of elementary analytic geometry, particularly with the use of those texts which set up a positive sense for all directions. For each statement it should be carefully indicated whether distances are meant in a directed (algebraic) sense or in a non-directed (geometric) sense. For example, contrast the two interpretations of the locus of point P equidistant from M and N:

$$\text{(a)} \quad MP=NP \qquad\qquad \text{(b)} \quad |MP|=|NP|.$$

For (a) the senses as well as the lengths must be equal, and in general certain parts of the perpendicular bisector of MN do not satisfy. When we write $[(x_1-x_2)^2+(y_1-y_2)^2]^{\frac{1}{2}}$, we automatically indicate absolute values, and when we square both sides we cancel out some of the effect

of signs in the original equation; hence we get an equation which corresponds to the interpretation (b). An algebraic expression which corresponds precisely to (a) is difficult to find.

No. 427. Proposed by *Isabella Burdick*, student, Missouri State Teachers College.

Find the length of the spiral $r\Theta = a$ from (r_1,Θ_1) to (r_2,Θ_2).

Solution by *Albert Farnell*, University of California.

Expressing the arc as a function of the angles:*

$$S = \int^{\Theta_2} \sqrt{(dr^2 + r^2 d\Theta^2)} = a \cdot \int_{\Theta_1}^{\Theta_2} \sqrt{(\Theta^2+1)} \cdot d\Theta/\Theta^2.$$

Letting $\Theta = \tan u$, we have formally:

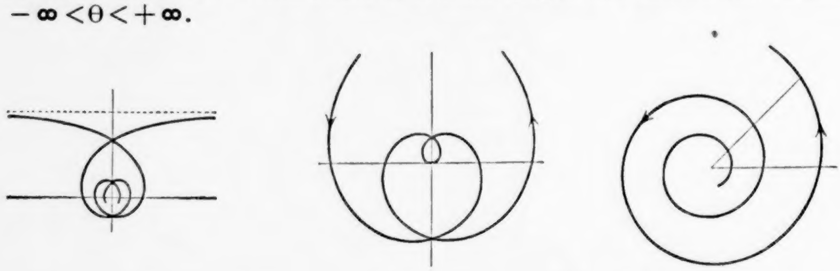
$$\int \sqrt{\Theta^2+1} \cdot d\Theta/\Theta^2 = \int (\sin^2 u + \cos^2 u) du/\sin^2 u \cos u$$

$$= \int du/\cos u + \int \cos u du/\sin^2 u$$

$$= ln(\sec u + \tan u) - \csc u + C.$$

Thus $$S = a \cdot [ln(\Theta + \sqrt{\Theta^2+1}) - \sqrt{\Theta^2+1}/\Theta]_{\Theta_1}^{\Theta_2}.$$

Editor's Note. The hyperbolic, archimedean, and logarithmic spirals with equations $r\Theta = a$, $r = a\Theta$, and $r = e^{a\Theta}$, respectively, are closely associated. With positive a, their graphs are shown for values $-\infty < \Theta < +\infty$.



The arrows indicate directions of tracing points as Θ increases. Evi-

*It is important that the limits Θ_1, Θ_2 have the same sign. If the integration be with respect to r instead, we may write $\sqrt{(1+a^2/r^2)} = \sqrt{(a^2+r^2)}/r$ only if r is positive.—ED.

dently, the hyperbolic and archimedean spirals are symmetrical with respect to the y-axis and intersect themselves at points along this axis. The hyperbolic spiral, composed of two branches, has the asymptote $y=a$ since $y=r\cdot\sin\Theta=(a\cdot\sin\Theta)/\Theta$ approaches a as Θ approaches O. Neither the hyperbolic nor logarithmic spiral passes through the origin.

Arc lengths for these curves in the order given are

$$\int\sqrt{(1+a^2/r^2)}\cdot dr,\qquad \int\sqrt{(1+r^2/a^2)}\cdot dr,\qquad \int\sqrt{(1+1/a^2)}\cdot dr.$$

From the first it is clear that the length from (r_1,Θ_1) along a branch of the hyperbolic spiral winding in toward the origin does not remain finite as r approaches O. Thus it would be improper to ask for the arc length between points on different branches. In the second, although an arc length does not remain finite as r approaches ∞, for any two selected points of the curve there corresponds a finite arc length. In the third, any arc length is proportional to the difference of the polar distances to the end points of the arc and has a finite limit as Θ approaches negative ∞.

Also solved by *W. N. Huff, C. D. Humphries, D. L. MacKay, P. D. Thomas, W. D. Wheatley, Marvin Wyman, M. I. Chernofsky,* and the *Proposer.*

No. 428. Proposed by *H. C. D. McClusky*, Colgate University.

An open box is to be formed from a rectangular piece of cardboard, a by b, by cutting out squares of side x at each corner and folding up the flaps. Find sets of integers, a and b, such that x is a rational number when the volume is a maximum.

Solution by the *Proposer.*

Assume $a>b$. It is seen that the volume, V, is

$$V=x(a-2x)(b-2x)$$

where $x<\frac{1}{2}b$. Then its derivative,

$$dV/dx=12x^2-4(a+b)x+ab,$$

will vanish when

$$x=[a+b-(a^2-ab+b^2)^{\frac{1}{2}}]/6,$$

where the $+$ sign before the radical is omitted because $a>b$ implies that the numerical value of $(a^2-ab+b^2)^{\frac{1}{2}}$ is greater than b, so that

$x > (a+2b)/6 > b/2$, contrary to the limitations on x. Now the value of x will be rational, if and only if, the expression $a^2 - ab + b^2$ is a square.

The problem of finding pairs of integers, a and b, such that $a^2 - ab + b^2$ is a square is well known. Neuberg's formulae (Dickson's *History of the Theory of Numbers*, v. 2, p. 405) state that if $a = r(2pq - q^2)$, $b = r(p^2 - q^2)$, and $z = r(p^2 - pq + q^2)$, then the relation $a^2 - ab + b^2 = z^2$ holds. To fit this problem with $a > b$, it is only necessary to take $2q > p$ and any number of solutions can be obtained.

Also solved by *W. H. Bradford, Morris I. Chernofsky, A. B. Farnell, C. N. Mills, D. L. MacKay,* and *P. D. Thomas.*

No. 429. Proposed by *Paul D. Thomas*, Southeastern State College, Oklahoma.

Construct a triangle given the radius of the nine-point circle and the lengths of an internal bisector and an exsymmedian issued from the same vertex.*

Solution by *D. L. MacKay*, Evander Childs High School, New York.

Let $O(R)$ be the circumcircle of triangle ABC, AT the angle bisector, AS the tangent to circle O at A cutting BC at S. Then AS is the exsymmedian.

Since $\angle STA = \angle(B + A/2) = \angle SAT$, $SA = ST$. Thus since SA is given, we may construct the isosceles triangle SAT. The circumradius R, equal to twice the radius of the nine-point circle, is known. Hence the perpendicular to AS at A equal to R yields point O and circle $O(R)$ cuts ST at B and C.

Also solved by the *Proposer* and *Walter B. Clarke* who mentions that the bisectors of the angles formed at the right angle by the altitude dropped upon the hypotenuse meet the hypotenuse in points whose distances from adjacent vertices equal the legs.

Note by *F. A. Foraker*, University of Pittsburgh.

Examples set for a class by the instructor are intended to afford practice in correct techniques. Care must be taken to avoid examples like the following in which the result would follow also from incorrect methods:

$$\sqrt{1+i+\sqrt{2i}} + \sqrt{1+i-\sqrt{2i}} = \sqrt{2+2i}. \tag{1}$$

$$\sqrt{1+i+\sqrt{2i}} - \sqrt{1+i-\sqrt{2i}} = \sqrt{2\sqrt{2i}}. \tag{2}$$

*These lengths are measured from the vertex to the points where the bisector and exsymmedian meet the opposite side of the triangle.—ED.

Observe that "obvious" cancellations under the radical signs still yield correct results.

Note by *G. W. Wishard*, Norwood, Ohio.

Sometimes it is erroneously stated that no more than three successive primes can be in arithmetical progression. (Cantor conjectured that three successive primes are not in arithmetical progression unless one of them is 3.) By inspection of Lehmer's *List of Prime Numbers* the writer found many examples of four successive primes in arithmetical progression. In each case the common difference is 6; the first terms are, respectively, 251, 1741, 3301, 5101, 5381, 6311, 13451, 14741, 15791, 15901, 17471, 18211, 23321, 26171.

Do readers have other interesting examples or fallacies to submit, perhaps without making formal problems of them?—ED.

PROPOSALS

No. 451. Proposed by *N. A. Court*, University of Oklahoma.

Two positive segments PP', QQ' move, in any manner whatever and independently of one another, on two fixed skew lines. The points L, M divide the segments PQ, $P'Q'$ in a given ratio, both in magnitude and in sign. Show that the direction of the line LM and the length of the segment LM are fixed.

No. 452. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Through a fixed point P in the plane of a given angle A draw the line that forms with the sides of the angle a triangle ABC such that $b+c-a$ is a minimum.

No. 453. Proposed by *W. V. Parker*, Louisiana State University.

Find all the points on the hyperbola

$$3x^2-y^2+2x-y=0$$

for which both coordinates are integers.

No. 454. Proposed by *James F. Callicott*, student, Colgate University.

Assuming that the curve $y=x^3+ax^2+bx+c$ has a maximum point at A and a minimum point at B, show that the points on the

curve where the curvature is greatest are located outside that part of the curve which is between A and B.

No. 455. Proposed by *H. T. R. Aude*, Colgate University.

If an integer N is written in the scale r using place value, the symbol zero and the other necessary $r-1$ symbols, it will require at least one of these symbols (digits). Let the sum of these digits be S_1, which is expressed in fewer digits than N. Let S_2 be the sum of the digits of S_1, and so on for S_3, $S_4, \cdots$, until only one digit is needed. Denote the last number by Σ. Thus, for 3457 in the scale of ten, $S_1=19$, $S_2=10$ and $\Sigma=1$. For 3457 in the scale of eight, $S_1=23$ and $\Sigma=5$.

1. Show that in the scale of nine, the sum of any number divisible by four and the number preceding it is a number whose Σ is seven.

2. Show that in the scale of seventeen, the sum of any four successive numbers, where the greatest is a multiple of four, is a number whose Σ is ten.

3. Show that in the scale of ten the sum of any three successive numbers, where the greatest is a multiple of three, is a number whose Σ is six.

No. 456. Proposed by *V. Thébault*, Tennie (sarthe-France).

In an orthocentric tetrahedron $ABCD$, inscribed in a sphere (O), the first twelve point sphere is (O_1) and the second twelve point sphere (O_2). Show that the lengths of the tangents to (O_1) and (O_2), drawn from an arbitrary point of (O), are in the ratio of $\sqrt{3}$ to 2.

No. 457. Proposed by *N. A. Court*, University of Oklahoma.

The line joining the center of the second twelve point sphere of an orthocentric tetrahedron to the mid-point of the segment joining the orthocenter to a vertex passes through the corresponding vertex of the twin tetrahedron.

Bibliography and Reviews

Edited by
H. A. SIMMONS and JOHN W. CELL

Fourier Series and Orthogonal Polynomials (The Carus Monograph Number Six). By Dunham Jackson. Open Court Publishing Company, Chicago, 1941. viii+234 pages. $2.00.

This Monograph measures up fully to the expectation of the reader who is acquainted (and who is not ?) with the lucid and elegant presentation—oral and written—of Dunham Jackson. As such, it is a valuable addition to the Carus Mathematical Monographs Series, designed to give to a person acquainted with the ordinary college courses in calculus *and differential equations* (an obvious omission in the Preface) the fundamentals of the theory and applications of orthogonal functions. Some readers will find the exposition interesting and stimulating *per se;* others will find this book a fine source to draw from in giving a course on "Differential Equations of Mathematical Physics". This enhances the timeliness and importance of Jackson's Monograph. We may add that it contains more than its title implies, namely, not only Fourier Series and orthogonal polynomials, but also Bessel Functions and Spherical Harmonics, also applications to various equations which one encounters in Mathematical Physics, including the Wave—Equation.

The treatment obviously cannot pretend to be complete. However, the proofs given are rigorous and simple, and the reader finds ample and lucid suggestions as to possible extensions and generalizations. Extensive and skillful use is made of the "broken line function".

The eleven chapters deal with Fourier Series, Legendre Polynomials, Bessel Functions, Boundary Value Problems, Double Fourier Series and Laplace Series (notably in application to Laplace Equation), Pearson's Frequency Functions, Orthogonal Polynomials in general, the polynomials of Jacobi, Hermite and Laguerre, and convergence of such series. More space is devoted—and with full justification—to the most important orthogonal functions, namely, Fourier Series, Legendre Polynomials and Bessel Functions. The book closes with a series of interesting "Exercises" and a bibliography.

A reviewer must always find shortcomings in the book he writes about in order to ward off any suspicion that he has *not* read it.

We note regretfully the lack of historical facts, especially when treating Fourier Series, which have played such an important part in the development of Analysis. It seems to us that in books of this type where the reader is very likely to get his *first* knowledge of the subject treated a historical background is of special interest (Cf. the admirable historical introduction in Bliss' Monograph). For the same reason the terminology should be amplified in places (this would afford more continuity for the reader who wishes to turn to more extensive literature). Thus, Parseval Theorem (p. 29) is generally known as "Parseval Formula". In treating Fejér's summation of Fourier Series (pp. 31-32), why not add "Césaro means (C,I)" and thus render well deserved tribute to a distinguished mathematician? Why not add the name of Laplace to the integral representing the Legendre polynomial $P_r(x)$ (p. 58)? When treating Laplace Series (pp. 121-126), why not introduce tne "Associated Legendre Functions"?

We further believe that the section on harmonic polynomials (pp. 126-138) could have been omitted. Instead, one would like to find in the chapter on Fourier Series the elegant result of Hurwiz concerning a relation between perimeter and area of closed curves—a simple application of the Parseval Formula.

A few minor remarks follow. In Chapter I (Fourier Series), one would like to read a little more about the expansion of functions defined in $(0, \pi)$ only (even and odd continuation); this is used in a later chapter dealing with the vibrating string. Section 9 (p. 17) should be started with the statement that *x is a fixed point in* $(-\pi,\pi)$. Sections 14, 15 should be headlined: Some properties and applications of Fourier expansion. On p. 45 the statement "... for sufficiently small values..." needs insertion of the word "numerically". In the treatment of Legendre Polynomials (Chapter II), it seems desirable to add Parseval Formula, also a discussion, if restricted, of convergence of the Legendre Series expansion at the end—points $\pm I$. On page 71, it is rather obscure how the identity $J_0''(t)+(1/t)J_0'(t)+J_0(t)=0$ is derived. On page 93 and elsewhere, choosing the sign of the constant when "separating variables" in a partial differential equation is not merely a matter "of convenience", as stated by the author; a more elaborate argument is desirable. Section 5 on page 101 should be headlined: Laplace equation (in plane) in polar coordinates. Chapter V should be headlined: Double *Fourier* Series..... The treatment of Laplace series (pp. 121-—) is somewhat confused due to an interruption caused by the necessary discussion of the Associated Legendre Functions (see above). Such interruption could have been avoided by introducing these functions earlier, namely in Chapter II, when dealing with Legendre Polynomials, to which they form a natural sequel.

The above critical remarks (some reflecting the personal taste of the reviewer) do not detract from the fine qualities of Jackson's Monograph, which, we feel certain, will render much service as a stimulating source of information on a very important branch of analysis.

The reviewer cheerfully pleads guilty to not having detected misprints in this beautifully printed book (but why not "Korous' Theorem" in place of "Korous's Theorem?, p. 125).

The University of Pennsylvania. J. SHOHAT.

A Survey of Modern Algebra. By Garrett Birkhoff and Saunders MacLane. The Macmillan Company, New York, 1941. xi+448 pages. $3.75.

This exposition of the elements of modern algebra has been planned with great skill, and the plan has been carried through very successfully. It is a unified and comprehensive introduction to modern algebra. The classical algebra is nicely embedded in this structure, as are also applications to other fields of thought.

This book is distinguished by its procedure from the concrete to the abstract. Familiar examples are carefully presented to illustrate each new term or idea which is introduced. Then the abstract definition appears simple, and the theoretical properties which are deduced from the definition exhibit the power of the concept.

This book is distinguished also by the great clarity with which all details have been presented. Statements of theorems and defintions are precisely phrased. Proofs are very carefully organized. Illustrations are penetratingly exhibited. Also, methods of procedure and general information are discussed as needed, when they are not easily and explicitly available elsewhere. The lists of graded, non-trivial exercises also serve to clarify the text.

The first five chapters give a postulational development of integers and integral domains, of rational numbers and fields, of real and complex numbers and rings. A wealth of material is included, ranging from divisibility of integers and de Moivre's theorem to the Peano postulates and Dedekind cuts.

The next five chapters, on abstract groups, linear spaces, matrices, linear groups, and determinants, constitute a central half of the book. A feature of the excellent chapter on abstract groups is the fact that only rarely is the discussion limited to finite groups. In the chapter on vector spaces, the emphasis on inner products is admirable. A noteworthy feature of the chapter on the algebra of matrices is the discussion of rectangular matrices. The long chapter on linear groups is especially commended. These chapters VI to X could be read independently of the preceding chapters.

A chapter on the algebra of classes, and a chapter on transfinite arithmetic, precede the final three chapters on rings and ideals, algebraic and transcendental extensions of a field and the Galois theory. The section on applications to algebraic geometry enhances the discussion of ideals. The section on adjunction of roots is outstanding in an excellent chapter on extension of fields.

Misprints were noted on pages 74, 126, 212, 284.

The printing and type are excellent.

? 1842

Northwestern University. L. W. GRIFFITHS.

A Treatise on Algebra. By George Peacock. Vol. I, Arithmetical Algebra. Reprinted from the 1942 edition. xvi+399 pages. Vol. II, On Symbolical Algebra and Its Applications to the Geometry of Position. Reprinted from the 1845 edition. x+455 pages. Scripta Mathematica, Yeshiva College, New York, 1940. $6.50.

Peacock's *Treatise on Algebra* enjoys the distinction of being named by Mortimer J. Adler in *How to Read a Book* in his chapter on *The Great Books.* It is mentioned along with Hilbert's *Foundations of Geometry* and Dedekind's *Theory of Numbers* as a book which even the layman might dare to read.

The book is outstanding for its clarity of exposition; for indeed Peacock was a teacher of the first class, a distinction rare among British contemporary mathematicians. Peacock did much to reform the teaching of algebra and to place it on a scientific basis through his *Treatise on Algebra,* both the earlier volume published in 1830 and the present treatise of 1842 and 1845. The significance of this work may be further appreciated when we realize that it was written at a time when books appeared in Britain actually protesting the use of negative numbers. What appears today quite commonplace to one who is not familiar with any algebra except the elementary type, was a novelty when Peacock published his work. Indeed we owe much to Peacock for our modern conception of algebra. It was he who developed algebra as an abstract system of symbols to be combined according to operations that conform with preassigned postulates, and dispensed with the then current idea that the symbols of algebra had to represent the numbers of ordinary arithmetic.

In his attempt to place algebra on a strictly logical basis, Peacock divided the science of algebra into two parts, *Arithmetical Algebra,* treated in Volume I, and *Symbolical Algebra* in Volume II.

He described *arithmetical algebra* as follows:

"In arithmetical algebra, we consider symbols as representing numbers, and the operations to which they are submitted as included in the same definitions (whether expressed or understood) as in common arithmetic: The signs + and − denote the operations of addition and subtraction in their ordinary meaning only, and those oper-

ations are considered as impossible in all cases where the symbols subjected to them possess values which would render them so, in case they were replaced by digital numbers: thus in expressions, such as $a+b$, we must suppose a and b to be quantities of the same kind: in others, like $a-b$, we must suppose a greater than b, and therefore homogeneous with it; in products and quotients, like ab and a/b, we must suppose the multiplier and divisor to be abstract numbers: all results whatsoever, including negative quantities, which are not strictly deducible as legitimate conclusions from the definitions of the several operations, must be rejected as impossible, or as foreign to the science."

Volume I treats of the principles of *arithmetical algebra;* the fundamental operations in arithmetic; extraction of the roots of numbers and the properties of surds; ratios and proportions; solution of equations, linear and quadratic, including simultaneous equations; progressions; permutations and combinations; binomial products and powers; solution of indeterminate equations of the first degree; and the symbolical representation and properties of numbers—all topics being treated with the restrictions imposed by arithmetical algebra.

Theory is explained in much detail and concise summary notes appear in the margins. As is typical of textbooks of Peacock's time, the book contains few applications and problems. Its greatest value is its historic importance in showing the development of modern algebra from its beginnings in arithmetical algebra. The teacher of arithmetic and elementary algebra should find this volume useful in giving greater significance to the fundamentals of arithmetic and algebra.

Volume II is devoted to *symbolical algebra* described by Peacock as follows: "*Symbolical algebra* adopts the rules of *arithmetical algebra*, but removes altogether their restrictions: thus symbolical subtraction differs from the same operation in arithmetical algebra in being possible for all relations of the value of the symbols or expressions employed."

The "*principle of the permanence of equivalent forms*" is made the foundation of symbolical algebra and is expressed as follows: "Whatever algebraic forms are equivalent, when the symbols are general in form but specific in value, will be equivalent likewise when the symbols are general in value as well as in form...... thus the product of a^m and a^n which is a^{m+n} when m and n are whole numbers, and therefore general in form though particular in value, will be their product likewise when m and n are general in value as well as in form."

The topics treated in Volume II are those of Volume I, extended to include negatives and imaginaries, and the further subjects of trigonometry, logarithms, series, partial fractions, solution of cubic and biquadratic equations, recurring equations, simultaneous equations and the theory of elimination, and binomial equations.

The terminology employed throughout the treatise is typical of the age in which it was written and in many cases differs from that employed in modern textbooks on college algebra and theory of equations. But the exposition is everywhere clear and the book possesses a mathematical elegance more characteristic of an original work than of a mere textbook. Volume II is recommended to all teachers of algebra.

MAY M. BEENKEN.

A Course of Analysis. By E. G. Phillips, Cambridge University Press, 2nd ed., 1939. viii+361 pages.

The first edition of this book was reviewed by H. M. Gehman in American Mathematical Monthly 38 (1931), pp. 166-168 and by E. W. Chittenden in Bulletin American

Mathematical Society 38 (1932), pp. 168-169. Both of these reviews pointed out the looseness of statement which occurred frequently in the theorems of the first edition. According to the author, the second edition undertakes chiefly to "clarify a few obscurities and to correct some errors", leaving the subject matter of the first edition substantially unchanged. A type of statement criticized by Gehman is one of the form:

If $g(x)$ is continuous in (a,b), then

$$\int_a^b g(x)h(x)dx = g(\xi)\int_a^b h(x)dx,$$

where ξ lies between a and b." This is a statement from the second edition p. 189, lines 9-12. It is one of the passages specifically cited by Gehman in this review which remain uncorrected in the second edition. In checking Gehman's list this reviewer found that several of the ambiguous statements cited by the earlier review had not been properly reworded. Several others had been corrected. It seems clear that the author did not make use of Gehman's painstaking review in preparing the second edition, and why not?

In the opinion of this reviewer however, the book remains one of the better texts in the difficult field which covers a student's transitional period between the study of elementary calculus and all out real variables. It presents the subject more as a finished body of knowledge to be learned than as a method of analysis in which the student seeks to acquire competency and then skill. Whether or not this is good is a matter of opinion.

A good feature of the book is the chapter on inequalities which is unusual and desirable in an elementary book. The treatment of the Riemann integral is well done. The subject of uniform convergence is developed in more detail than is usually to be found in books of this type.

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A Diagnostic Study of Students' Difficulties in General Mathematics in First Year College Work. By Elizabeth N. Boyd. Bureau of Publications, Teachers College, Columbia University, New York, 1940. 152 pages. $1.75.

Testing in mathematics for teaching and diagnostic purposes has been used in increasing amounts on the elementary and secondary levels, "but thus far no diagnostic tests in general mathematics for college freshmen have been published." Dr. Boyd undertook her study because she believed that "an urgent need for diagnostic work in the field of college freshman mathematics seems to exist" and that such tests for the diagnosis of errors would lead to the improvement of instruction.

Dr. Boyd accepted the course in general mathematics consisting of topics in trigonometry, analytic geometry, and the function and its derivative as presented to college freshmen at Hunter College, the women's college of the City of New York. A series of twenty ten-minute objective tests were prepared to cover most of the topics in the course and were administered to classes taught by the author and a colleague after the material had been taught. The study has been limited mainly to tests of information and skill, although there is a recognition of the importance of tests of ability to reason, appreciation, organization, and sustained attention in attacking a problem. The results were analyzed according to errors in reading, expression, and symbolism; in skill in algebra and reading tables; in memory; in concepts and principles; in relationships,

generalization, and significance; in method of attack; in inability to use resources; and in processes involving several steps. Most of the book is devoted to a discussion of these errors and some remedial work.

In the discussion, each of the test questions is presented according to the classification of errors. The questions are brief and carefully prepared for clear understanding. The per cent of students who failed each question and the specific reasons for the failures are given. The reader will find the discussion well organized for interesting reading, particularly if the appendices are studied first. Perhaps some of the material in the appendices could have been included advantageously in the main body of the study. Experienced teachers of college freshman mathematics will not find the results startling, but they will appreciate the objective evidence. Inexperienced teachers will find the results revealing and worthy of careful study as an aid to the organization of their teaching. The author's accomplishments with remedial teaching are encouraging.

Throughout the book, Dr. Boyd continually demonstrates envisioning the significance of her study by interpreting the results from the diagnosis of errors for the purpose of improving the teaching of mathematics. The last pages of the final chapter are devoted to suggestions to instructors for improvement in classroom practice through attention to the findings of the study.

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In some way or other, openly or hidden, even under the most uncompromising formalistic, logical, or postulational aspect, constructive intuition always remains the most vital element in mathematics.—From *What is Mathematics?* by Richard Courant and Herbert Robbins, p. 88.